

# Methods of Mathematical Physics

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# Introduction

These notes are based on the course “Methods of Mathematical Physics” given by Dr. M. G. Worster in Cambridge in the Michæmas Term 1997. These typeset notes are totally unconnected with Dr. Worster. Recommended books will be discussed at the end.

Other sets of notes are available for different courses. At the time of typing these courses were:

Probability	Discrete Mathematics
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Geometry	Dynamics of D.E.'s
Foundations of QM	Electrodynamics
Methods of Math. Phys	Fluid Dynamics 2
Waves (etc.)	Statistical Physics
General Relativity	Dynamical Systems
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# Chapter 1

## Complex Variables

### 1.1 Conventions

Various people use different meanings for analytic, regular, etc. We will use these:

**Definition.** A function is analytic at a point iff there exists an open neighbourhood of the point in which the function is complex differentiable. This is true iff the function has a Taylor expansion about that point.

**Definition.** A function is analytic in a domain iff it is analytic at every point in the domain and single valued in the domain.

**Definition.** A function is singular at a point iff it is not analytic at the point.

**Definition.** A function has an isolated singularity at a point iff it is analytic in some punctured ball about the point, or iff it has a Laurent expansion about the point.

### 1.2 Cauchy Principal Value

The integral  $\int_{-1}^2 \frac{dx}{x}$  does not exist. If we consider

$$I(\eta, \xi) = \int_{-1}^{\eta} \frac{dx}{x} + \int_{\xi}^2 \frac{dx}{x} = \log \frac{2\eta}{\xi}$$

then we see that  $\lim_{\eta, \xi \rightarrow 0} I(\eta, \xi)$  can be made to do anything we want. The particular choice  $\eta = \xi$  gives a limit of  $\log 2$ , and this is the Cauchy principal value of the original integral. More formally:

If  $f(x)$  has a *simple* pole at  $x = c$  with  $a < c < b$  then the Cauchy principal value of  $\int_a^b f(x) dx$  is defined to be

$$\lim_{\epsilon \rightarrow 0} \left[ \int_a^{c-\epsilon} f(x) dx + \int_{c+\epsilon}^b f(x) dx \right].$$

It is written  $\mathcal{P} \int_a^b f(x) dx$ . For instance,  $\mathcal{P} \int_{-1}^1 \frac{dx}{x} = 0$ .

Consider the complex contour  $\Gamma$  shown and let  $I = \int_{\Gamma} f(z) dz$ , and let  $f$  be analytic except for a simple pole at  $c$ .

By Cauchy's theorem,  $\int_{\Gamma} = \int_{\Gamma'}$ . In the limit  $\epsilon \rightarrow 0$ , we get

$$\int_{\Gamma'} f(z) dz = \mathcal{P} \int_a^b f(x) dx + \lim_{\epsilon \rightarrow 0} \int_{C_{\epsilon}} f(z) dz.$$

On  $C_{\epsilon}$   $z = c + \epsilon e^{i\theta}$  for  $\pi < \theta < 2\pi$ . Since  $f$  has only a simple pole at  $z = c$  we get  $f(z) = \frac{\text{Res}}{z-c} + a_0 + \dots$ . Then  $\lim_{\epsilon \rightarrow 0} \int_{C_{\epsilon}} f(z) dz = \pi i \text{Res}$  by Cauchy's theorem. Thus finally

$$\int_{\Gamma} f(z) dz = \mathcal{P} \int_a^b f(x) dx + \pi i \text{Res},$$

where as the name suggests, Res is the residue of  $f$  at  $z = c$ . Similarly, going the other way round,

$$\int_{\Gamma} f(z) dz = \mathcal{P} \int_a^b f(x) dx - \pi i \text{Res}.$$

We can extend this idea to more general complex contours, such as

to get

$$\int_{\Gamma_1} f(z) dz = \mathcal{P} \int_{\Gamma} f(z) dx - \pi i \text{Res} \quad \text{and} \quad \int_{\Gamma_2} f(z) dz = \mathcal{P} \int_{\Gamma} f(z) dx + \pi i \text{Res}.$$

**Example.** Find

$$\int_{\Gamma} \cot z dz.$$

*Solution.*

$$\begin{aligned} \int_{\Gamma} \cot z dz &= \mathcal{P} \int_{-\infty}^{\infty} \cot x dx - \pi i \\ &= 0 - \pi i \quad \text{by symmetry.} \end{aligned}$$

□

**Example.** Find

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx.$$

*Method 1.* As the integrand is analytic we can deform the real axis into the contour  $\Gamma$ , thus

$$\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx = \int_{\Gamma} \frac{1 - \cos z}{z^2} dz.$$

Now we consider  $\int_{\Gamma \cup C_R} \frac{1 - e^{iz}}{z^2} dz$  and take the real part. This integral is  $2\pi i \text{Res}$ .  $\square$

*Method 2.* We consider

$$\begin{aligned} \int_{\Gamma} \frac{1 - e^{iz}}{z^2} dz &= \mathcal{P} \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} dx - i\pi \text{Res} \\ &= \mathcal{P} \int_{-\infty}^{\infty} \frac{1 - e^{ix}}{x^2} dx - \pi = 0. \end{aligned}$$

Thus  $\mathcal{P} \int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx = \pi$ , but this is the actual integral  $\int_{-\infty}^{\infty} \frac{1 - \cos x}{x^2} dx$  as this has no singularity at 0.  $\square$

### Singularities at Infinity

If the integral diverges at  $\infty$  define

$$\mathcal{P} \int_{-\infty}^{\infty} f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx.$$

For instance,  $\mathcal{P} \int_{-\infty}^{\infty} \frac{dx}{x-i} = \lim_{R \rightarrow \infty} \log \frac{R-i}{-R-i} = i\pi$  on the principal branch.

**Theorem.** The function  $f(z) = \int_{t_1}^{t_2} g(z, t) dt$  is analytic in some domain  $\mathcal{D} \subset \mathbb{C}$  if  $g(z, t)$  is analytic in  $z$  for each  $t \in (t_1, t_2)$ . Furthermore  $\frac{df}{dz} = \int_{t_1}^{t_2} \frac{\partial g}{\partial z}(z, t) dt$ .

*Proof.* Omitted; see Copson page 108.  $\square$

If either  $t_1$  or  $t_2$  is infinite then the convergence of the integral must be uniform for  $z \in \mathcal{D}$ . This result extends to

$$f(z) = \int_{\Gamma} g(z, \zeta) d\zeta$$

simply by parametrizing  $\Gamma$ .

## 1.3 Analytic Continuation

**Theorem.** Suppose  $\mathcal{D}_1$  and  $\mathcal{D}_2$  are disjoint simply connected domains which share a piece of common boundary  $\Gamma$ , with  $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \Gamma$  simply connected as well. Let  $f_1(z)$  be analytic on  $\mathcal{D}_1$  and continuous on  $\mathcal{D}_1 \cup \Gamma$  and similarly let  $f_2(z)$  be analytic on  $\mathcal{D}_2$  and continuous on  $\mathcal{D}_2 \cup \Gamma$ . Suppose further that  $f_1 = f_2$  on  $\Gamma$ . Then if we define

$$g(z) = \begin{cases} f_1(z) & z \in \mathcal{D}_1 \cup \Gamma \\ f_2(z) & z \in \mathcal{D}_2 \end{cases}$$

$g$  is analytic on  $\mathcal{D}$ .  $g$  is called the analytic continuation of  $f_1$  from  $\mathcal{D}_1$  into  $\mathcal{D}$ .

*Proof.* Consider  $I = \oint_C g(z) dz$  for some contour  $C \subset \mathcal{D}$ . Now  $I = 0$  if  $C \subset \mathcal{D}_1$  or  $C \subset \mathcal{D}_2$ , so we just need to consider the sketched case. Then  $\oint_C = \oint_{C_1} + \oint_{C_2} = 0$  and thus  $g$  is analytic by Morera's theorem.  $\square$

The analytic continuation is unique (if it exists) by the following theorem.

**Theorem.** *If  $f$  is analytic in  $\mathcal{D}$  and has an infinite sequence of zeroes with a limit point in  $\mathcal{D}$  then  $f \equiv 0$  on  $\mathcal{D}$ .*

*Proof.* Let the limit point be at  $a$ ; then  $f(a) = 0$  by continuity. Either  $f \equiv 0$  or  $f(z) = (z-a)^m \phi(z)$  with  $\phi$  analytic and  $\phi(a) \neq 0$ . Now  $\phi$  is continuous and so there is a neighbourhood of  $a$  on which  $\phi \neq 0$ . Thus there exists a neighbourhood of  $a$  on which  $f$  is nonzero, giving a contradiction.  $\square$

If  $g_1$  and  $g_2$  are both analytic continuations then  $g_1 - g_2 = 0$  on  $\mathcal{D}_1$  and so  $g_1 \equiv g_2$  on  $\mathcal{D}$ .

### Continuation of power series

Suppose (for instance) that by hook or by crook we have obtained the power series expansion for  $f(z) = \sum_n z^n$ . This is analytic in  $|z| < 1$ . Then we can form a new series by Taylor expansion about some other point  $z_0$  such that  $|z_0| < 1$ . Hopefully this new power series has a convergent circle part of which is outside the original domain. We can continue this process to try to cover  $\mathbb{C}$ , but we may run into singularities or branch cuts.

### Functions defined by integrals

Suppose we have

$$f(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z-t} dt$$

defined for  $\Im z \neq 0$ . Can we find an analytic continuation of  $f_1(z) = f(z)$  for  $\Im z > 0$  into  $\mathbb{C}$ ? Define  $g(z) = \int_{\Gamma} \frac{e^{-\zeta^2}}{z-\zeta} d\zeta$ , with  $\Gamma$  chosen to lie below  $\zeta = z$ . Then  $g(z)$  is analytic.

1. If  $\Im z > 0$  we can deform  $\Gamma$  into  $\mathbb{R}$  to get  $g(z) = f_1(z)$ .
2. If  $\Im z = 0$  we get  $g(z) = \mathcal{P} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z-t} dt + \pi i \text{Res}$ .
3. If  $\Im z < 0$  we use the  $\Gamma$  sketched to get  $g(z) = f(z) + 2\pi i \text{Res}$ .

There are functions defined by integrals which we cannot continue, for example

$$f(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{z^2 + t^2} dt$$

cannot be continued from  $\Im z > 0$  into  $\mathbb{C}$  — there are two “pinching” singularities which prevent deformation of the contour as above.

## 1.4 Multivalued functions

The usual example is:  $f(z) = z^{\frac{1}{2}}$ . If we take  $z = re^{i\theta}$  then  $f(z) = \sqrt{r}e^{i\frac{\theta}{2}}$ . If we trace  $f(z)$  as  $z$  moves around a closed curve not encircling the origin we find that  $\theta$  returns to its original value and  $f$  is continuous. If the curve encircles the origin then  $\theta$  increases to  $2\pi$  and  $f(z) \rightarrow -\sqrt{r}e^{i\theta}$ .  $f$  is singular at  $z = 0$  — it is not analytic because it is not single valued in any neighbourhood of 0.  $f(z)$  has neither a Laurent expansion nor a residue at 0. 0 is called a branch point.

$f$  has a Taylor expansion about  $z = 1$  (for example) with circle of convergence  $|z - 1| < 1$ . We can extend the Taylor expansion by analytic continuation.

The continued function is discontinuous across a curve (or ray) from the origin to infinity. This curve is called a *branch cut*, and  $f$  is continued analytically to a simply-connected domain which excludes the branch cut.

Another favourite example is  $f(z) = \log z = \log r + i\theta$ . Now  $\Im f$  increases by  $2\pi$  on any trip around the origin and so  $z = 0$  is a branch point.

A slightly more complicated example is  $f(z) = (z^2 - 1)^{\frac{1}{2}}$  which has branch points at  $z = \pm 1$ . A useful way of doing it if we want  $f$  in the neighbourhood of the origin is:

but if we want to consider  $|z| \gg 1$  we may prefer to send both cuts away on the negative real axis as shown on the left.

It is easy to see that this is equivalent to the branch cut from  $-1$  to  $+1$  shown on the right (which is why it is useful for  $|z|$  large).

### 1.4.1 Branch cut integrals

These can be considered simplest by example. We thus look at

$$I = \oint_C (z^2 - 1)^{\frac{1}{2}} dz,$$

where  $C$  is any closed curve encircling the origin outside  $|z| = 1$ .

We introduce a branch cut on the real axis between  $\pm 1$ , and as the integrand is analytic in the cut plane we can deform the contour onto the cut. It is easy to see that  $\int_{\Gamma_{\epsilon_1}} \rightarrow 0$  and  $\int_{\Gamma_{\epsilon_2}} \rightarrow 0$ . By fiddling some more we get that  $f(x) = (1 - x^2)^{\frac{1}{2}} e^{\frac{i\pi}{2}}$  on  $\Gamma_1$  and  $f(x) = (1 - x^2)^{\frac{1}{2}} e^{-\frac{i\pi}{2}}$  on  $\Gamma_2$ . Thus

$$I = (e^{-\frac{i\pi}{2}} - e^{\frac{i\pi}{2}}) \int_{-1}^1 (1 - x^2)^{\frac{1}{2}} dx = -\pi i.$$

Another example, where we deliberately introduce a branch cut, is  $I = \int_0^\infty f(x) dx$  with  $f$  not even. We consider  $\int_0^\infty f(x) \log x dx$  as follows:

If  $f$  is sufficiently nice then  $\int_{C_R} \rightarrow 0$  as  $R \rightarrow \infty$ . We have

$$\int_{\Gamma_1} = \int_0^\infty f(x) \log x dx \quad \text{and} \quad \int_{\Gamma_2} = - \int_0^\infty f(x) (\log x + 2\pi i) dx.$$

Adding these two gives

$$\int_0^\infty f(x) dx = - \sum \text{Res} (f(z) \log z).$$

### 1.4.2 Riemann surfaces

We consider  $f_1(z) = z^{\frac{1}{2}}$  with  $f_1(x) = \sqrt{x}$  for  $x \in \mathbb{R}$  positive and  $f_2(z) = z^{\frac{1}{2}}$  with  $f_1(x) = -\sqrt{x}$  for  $x \in \mathbb{R}$  positive. We continue  $f_1$  around the origin from the positive real axis. At  $z = re^{i\pi}$  we can continue  $f_1$  onto a copy of the complex plane where it becomes  $f_2$ . If we follow  $f_2$  around again until its branch cut on the copy of the negative real axis we find that we can jump back onto our original complex plane. We have a function which is analytic everywhere in an enlarged space with two ‘‘Riemann sheets’’. Closed curves in this space encircle the origin an even number of times.

Another example is  $f(z) = \log z$  which has Riemann sheets in the form of an infinite spiral ramp.

## Chapter 2

# Special Functions

This chapter deals mainly with the gamma function and its relatives. Other special functions are encountered in the next chapter.

### 2.1 The Gamma Function

This is an analytic continuation of the factorial function from the positive integers into  $\mathbb{C}$ . We define

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt. \quad (2.1)$$

This integral is well defined for  $\Re z > 0$ . For  $\Re z > 1$  we integrate by parts to get the recurrence  $\Gamma(z) = (z-1)\Gamma(z-1)$  which we use to continue  $\Gamma$  into  $\Re z \leq 0$ . This continuation is analytic except for simple poles at the negative integers.

By straight integration we see that  $\Gamma(1) = 1$  and so by iteration,  $\Gamma(n+1) = n!$  for  $n$  a positive integer.

Note that (by change of variable in (2.1)).

$$\Gamma(m) = 2 \int_0^{\infty} x^{2m-1} e^{-x^2} dx. \quad (2.2)$$

## 2.2 The Beta function

We define

$$B(m, n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt. \quad (2.3)$$

This is well-defined for  $\Re m, \Re n > 0$ . We now derive a formula for the beta function in terms of the gamma function, using equation (2.2).

$$\Gamma(m)\Gamma(n) = 4 \int_0^\infty \int_0^\infty x^{2m-1}y^{2n-1}e^{-x^2+y^2} dx dy$$

Changing to polar co-ordinates we obtain

$$\begin{aligned} \Gamma(m)\Gamma(n) &= 4 \int_0^\infty r^{2(m+n)-1}e^{-r^2} dr \int_0^{\frac{\pi}{2}} \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta \\ &= \Gamma(m+n) \int_0^{\frac{\pi}{2}} 2 \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta. \end{aligned}$$

Putting  $\tau = \cos^2 \theta$  gives

$$\begin{aligned} \Gamma(m)\Gamma(n) &= \Gamma(m+n) \int_0^1 \tau^{m-1}(1-\tau)^{n-1} d\tau \\ &= \Gamma(m+n)B(m, n). \end{aligned}$$

Thus we have the required formula,

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (2.4)$$

and another integral representation of the beta function

$$B(m, n) = \int_0^{\frac{\pi}{2}} 2 \cos^{2m-1} \theta \sin^{2n-1} \theta d\theta. \quad (2.5)$$

### Special cases

Putting  $m = n = \frac{1}{2}$  into (2.4) and (2.5) we get  $\pi = \Gamma(\frac{1}{2})^2$ , which ought to be familiar, although perhaps not in quite this form.

If  $m = z$  and  $n = 1 - z$  we require  $0 < \Re z < 1$  for convergence of (2.3). With this restriction we evaluate  $B(z, 1 - z) = \Gamma(z)\Gamma(1 - z)$ .

$$\Gamma(z)\Gamma(1 - z) = \int_0^1 t^{z-1}(1-t)^{-z} dt = I \quad \text{say.}$$

Putting  $t = \frac{1}{1+s}$  we get

$$I = \int_0^\infty \frac{s^{-z}}{1+s} ds.$$

Evaluating this as a branch cut integral gives

$$I(1 - e^{-2\pi iz}) = 2\pi i e^{-\pi iz}$$

and hence

$$I = \frac{\pi}{\sin \pi z}.$$

Thus  $B(z, 1 - z) = \frac{\pi}{\sin \pi z}$  for  $0 < \Re z < 1$ . But this formula is analytic on  $\mathbb{C} \setminus \mathbb{Z}$  and so by analytic continuation

$$\Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z} \quad (2.6)$$

on  $\mathbb{C} \setminus \mathbb{Z}$ . In fact it is true on the integers as well — if interpreted correctly!

We now want  $B(z, z)$  with  $\Re z > 0$ . (2.3) gives

$$\begin{aligned} B(z, z) &= \int_0^1 (t - t^2)^{z-1} dt \\ &= 2 \int_{\frac{1}{2}}^1 (t - t^2)^{z-1} dt, \end{aligned}$$

to avoid a branch cut on putting  $s = (2t - 1)^2$

$$\begin{aligned} &= 2^{1-2z} \int_0^1 s^{-\frac{1}{2}} (1 - s)^{z-1} ds \\ &= 2^{1-2z} B\left(\frac{1}{2}, z\right). \end{aligned}$$

We relate this to the gamma function using (2.4) to get *Legendre's duplication formula*

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \pi^{\frac{1}{2}} 2^{1-2z} \Gamma(2z). \quad (2.7)$$

Now we do

$$\begin{aligned} B(z, n + 1) &= \frac{\Gamma(z)\Gamma(n + 1)}{\Gamma(z + n + 1)} = \int_0^1 t^{z-1} (1 - t)^n dt \\ &= n^{-z} \int_0^n \tau^{z-1} \left(1 - \frac{\tau}{n}\right)^n d\tau. \end{aligned}$$

We take the limit as  $n \rightarrow \infty$  to get

$$\lim_{n \rightarrow \infty} \frac{\Gamma(z)\Gamma(n + 1)n^z}{\Gamma(z + n + 1)} = \int_0^\infty \tau^{z-1} e^{-\tau} d\tau = \Gamma(z). \quad (2.8)$$

We can rearrange this to get *Euler's limit* for the gamma function:

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n^z n!}{z(z + 1) \dots (z + n)}, \quad (2.9)$$

which can be thought of as showing the poles at  $\{0, -1, -2, \dots\}$ . We can use (2.8) to get

$$\lim_{n \rightarrow \infty} \frac{n! n^z}{(n+z)!} = 1. \quad (2.10)$$

We introduce the Hankel contour shown here.

Consider  $I(z) = \int_C e^{t-t^z} dt$ , which is also written  $\int_{-\infty}^{(0+)}$ .

$$\begin{aligned} \int_{\Gamma_1} &= \int_{\infty}^0 e^{-r} r^{-z} e^{i\pi z} e^{-i\pi} dr \\ &= e^{i\pi z} \int_0^{\infty} e^{-r} r^{-z} dr \\ \int_{\Gamma_2} &= -e^{i\pi z} \int_0^{\infty} e^{-r} r^{-z} dr. \end{aligned}$$

$\int_{\Gamma_\epsilon} \sim \epsilon^{1-z}$  which tends to zero if  $\Re z < 1$ . Thus

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{-\infty}^{(0+)} e^{t-t^z} dt. \quad (2.11)$$

We proved this for  $\Re z < 1$ , but in fact the derivative of (2.11) exists for all  $z$  and so  $\frac{1}{\Gamma(z)}$  is analytic and (2.11) holds for all  $z$ .

### 2.3 The Riemann zeta function

The Riemann zeta function is defined for  $\Re z > 1$  by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}. \quad (2.12)$$

Some “famous” results are  $\zeta(2) = \frac{\pi^2}{6}$  and  $\zeta(4) = \frac{\pi^4}{90}$ . We use the Hankel representation of the gamma function (2.11) to get

$$\begin{aligned} \frac{2\pi i n^{-z}}{\Gamma(1-z)} &= \int_{-\infty}^{(0+)} e^{n\tau} \tau^{z-1} d\tau \quad \text{and so} \\ \zeta(z) &= \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0+)} \frac{\tau^{z-1}}{e^{-\tau} - 1} d\tau. \end{aligned}$$

Thus  $\zeta(z)$  can only be singular at  $z = 1, 2, \dots$ , thus the only singularity is at  $z = 1$ . We therefore have the analytic continuation of  $\zeta$ :

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i} \int_{-\infty}^{(0+)} \frac{\tau^{z-1}}{e^{-\tau} - 1} d\tau. \quad (2.13)$$

Something which we do not prove is the reflection formula

$$\zeta(1-z) = 2^{1-z} \pi^{-z} \cos\left(\frac{1}{2}\pi z\right) \zeta(z) \Gamma(z), \quad (2.14)$$

which shows us that  $\zeta(z) = 0$  at  $z = 2n + 1$  for  $n = 1, 2, \dots$ .

By noting that  $2^{-z}\zeta(z) = \frac{1}{2^z} + \frac{1}{4^z} + \dots$  we see that  $(1-2^{-z})\zeta(z) = \frac{1}{1^z} + \frac{1}{3^z} + \dots$ . Continuing this process with the rest of the primes we obtain the Euler product for  $\zeta$ :

$$\zeta(z) = \prod_{i=1}^{\infty} (1 - p_m^{-z})^{-1}, \quad (2.15)$$

where  $p_m$  is the  $m^{\text{th}}$  prime. This is the reasoning behind the following (starred) section.

### 2.3.1 Applications to number theory

Let  $\pi(x)$  be the number of primes less than or equal to the real number  $x$ . Then from the Euler product (2.15) we have

$$\begin{aligned} \log \zeta(z) &= - \sum_{m=1}^{\infty} \log(1 - p_m^{-z}) \\ &= - \sum_{m=2}^{\infty} (\pi(m) - \pi(m-1)) \log(1 - m^{-z}) \\ &= - \sum_{m=2}^{\infty} \pi(m) [\log(1 - m^{-z}) - \log(1 - (m+1)^{-z})] \\ &= \sum_{m=2}^{\infty} \pi(m) \int_m^{m+1} \frac{z}{x(x^z - 1)} dx \quad \text{and so} \\ z^{-1} \log \zeta(z) &= \int_2^{\infty} \frac{z\pi(x)}{x(x^z - 1)} dx. \end{aligned}$$

This looks like some kind of transform of  $\pi(x)$ . We will see later that we can find approximations of  $\pi(x)$  from the locations of the singularities of  $\log \zeta(z)$  (or zeroes of  $\zeta(z)$ ).

In 1890 Hadamard proved that  $\zeta$  has no zeroes on  $\Re z = 1$ , which was enough to prove the *Prime Number Theorem*, that  $\pi(x) \sim \frac{x}{\log x}$  as  $x \rightarrow \infty$ . The Riemann Hypothesis of 1860 that the only zeroes of  $\zeta(z)$  in  $0 < \Re z < 1$  are on  $\Re z = \frac{1}{2}$  is enough to prove the stronger result

$$\pi(x) = \int_0^x \frac{dt}{\log t} + \mathcal{O}(x^{\frac{1}{2}} \log x),$$

the snag being merely that the Riemann hypothesis is still a hypothesis.



## Chapter 3

# Second order linear differential equations

We shall discuss equations of the general form

$$w'' + p(z)w' + q(z)w = 0. \quad (3.1)$$

The form of the solutions of this equation can be determined by the location and nature of the singularities of  $p$  and  $q$  in  $\mathbb{C}$ .

### Ordinary points

$z_0$  is an ordinary point (or regular point) of (3.1) if  $p$  and  $q$  are both analytic at  $z_0$ . The behaviour of  $w$  near  $z = z_0$  is determined by the leading order terms of the Taylor expansions of  $p$  and  $q$  about  $z_0$ . If  $p = \sum p_n(z - z_0)^n$  and  $q = \sum q_n(z - z_0)^n$ , and either  $p_0 \neq 0$  or  $q_0 \neq 0$  then we get

$$w'' + p_0 w' + q_0 w \sim 0. \quad (3.2)$$

This has solutions  $e^{m(z-z_0)}$ , which is analytic at  $z = z_0$ . This is enough to show that the solution of (3.1) is analytic, and carries over to cases when  $p_0 = q_0 = 0$ , and shows that  $w = \sum a_n(z - z_0)^n$ . We can determine the coefficients  $a_n$  by substitution, and the series converges at least out to the nearest singularity of  $p$  or  $q$  in  $\mathbb{C}$ . We can see this with Legendre's equation of order 1

$$(1 - z^2)w'' - 2zw' + 2w = 0. \quad (3.3)$$

We see that  $p$  and  $q$  both have singularities at  $z = \pm 1$ , but are analytic at  $z = 0$ . We expand  $w$  about  $z = 0$ , and equating coefficients gives  $a_n = \frac{n-3}{n-1}a_{n-2}$ . We thus get two series solutions, one of which terminates:

$$\begin{aligned} w &= a_0 \left(1 - z^2 - \frac{1}{3}z^4 + \dots\right) & \text{and} \\ w &= a_1 z. \end{aligned}$$

The series has (unsurprisingly) a radius of convergence 1.  $p$  and  $q$  are both singular at  $z = \pm 1$ , but we wish to know if we can analytically continue the series around these singularities.

### Singular points

$z = a$  is a singular point if either  $p$  or  $q$  is singular at  $z = a$ . We restrict to isolated singularities (when  $p$  and  $q$  have Laurent expansions).

Let  $z = a$  be an isolated singularity and choose  $R$  such that  $p$  and  $q$  are analytic in  $\mathcal{D} = \{z \in \mathbb{C} : 0 < |z - a| < R\}$ . Let  $\mathcal{C}$  be the circle  $\{z \in \mathbb{C} : |z - a| = \rho = \frac{R}{2}\}$  and take  $z_0 \in \mathcal{C}$ . We can construct two independent solutions (by series substitution, say)  $w_1$  and  $w_2$  about  $z = z_0$ , which both have radius of convergence  $\rho$ . We then choose  $z_1 \in \mathcal{C}$  with  $|z_1 - z_0| < \rho$ . Repeat (about 8 times) until we get back to a circle containing  $z_0$ . We have obtained solutions  $w_1^*$  and  $w_2^*$ , which are linear combinations of  $w_1$  and  $w_2$ :

$$\begin{pmatrix} w_1^* \\ w_2^* \end{pmatrix} = (\alpha_{ij}) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \quad (3.4)$$

The matrix  $(\alpha_{ij})$  is called the continuation matrix, which must be invertible, as we can continue the solutions backwards. We now examine the eigenvalues of  $(\alpha_{ij})$  to see what happens.

We first consider the distinct eigenvalue case, say  $\lambda_1$  and  $\lambda_2$ . Therefore,  $w_1^* = \lambda_1 w_1$  and  $w_2^* = \lambda_2 w_2$ . We write  $\lambda_i = e^{2\pi i \sigma_i}$ , and write  $w_k(z) = (z - a)^{\sigma_k} v_k(z)$ . Then  $w_k^* = \lambda_k (z - a)^{\sigma_k} v_k(z)$ . Thus  $v_k$  is single-valued around the circle, and therefore has at worst an isolated singularity at  $a$ , and so has a Laurent expansion. If the Laurent expansion terminates below then we can write

$$w_k(z) = (z - a)^{\sigma_k} \sum_{n=0}^{\infty} c_{n,k} (z - a)^n$$

(redefining  $\sigma_k$  if necessary). This is a *Frobenius expansion* and in this case we call  $z = a$  a regular singular point.

If we have two identical eigenvalues,  $\lambda_1 = \lambda_2 = \lambda$  (say), there are two distinct cases. If we can diagonalise  $(\alpha_{ij})$  then the results above hold. If we can't diagonalise  $(\alpha_{ij})$  then we can put  $(\alpha_{ij})$  in a Jordan Normal Form

$$(\alpha_{ij}) = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}.$$

Then the analysis for  $w_1$  is as before, and we look for  $w_2(z) = u(z)w_1(z)$ . Then  $u^* w_1^* = (1 + \lambda u) w_1$ . So we write  $u = \frac{\lambda^{-1}}{2\pi i} \log(z - a) + s(z)$ , so  $s(z)$  is single-valued and has a Laurent expansion. Putting all this together we get

$$w_2 = \frac{\lambda^{-1}}{2\pi i} (z - a)^{\sigma_1} (v_1 \log(z - a) + v_2(z)),$$

where  $v_1$  and  $v_2$  both have Laurent expansions.

## 3.1 Method of Frobenius

**Theorem.** *If  $z = 0$  is a singular point of (3.1) then it is a regular singular point (Laurent expansions terminate below) iff  $zp(z)$  and  $z^2q(z)$  are both analytic at  $z = 0$ .*

The method of Frobenius is to propose an infinite series  $w(z) = \sum_n a_n z^{\sigma+n}$ . We substitute this into (3.1) and take the coefficient of the lowest power of  $z$ . This is the indicial equation which determines two values of  $\sigma$ ,  $\sigma_1$  and  $\sigma_2$  say.

If  $\sigma_1 - \sigma_2 \notin \mathbb{Z}$  then we have two Frobenius series for  $w$ .

If  $\sigma_1 = \sigma_2$  then we must insert a logarithm,

$$w(z) = z^\sigma \left( (A + B \log z) \sum_{n=0}^{\infty} a_n z^n + B \sum_{n=0}^{\infty} b_n z^n \right).$$

If we have  $\sigma_1 - \sigma_2$  a positive integer then there is always a Frobenius series  $z^{\sigma_1} \sum_{n=0}^{\infty} a_n z^n$ . Either (by some miracle),  $z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n$  is also a solution, or we need to insert a logarithm to get

$$w_2(z) = z^{\sigma_1} (A + B \log z) \sum_{n=0}^{\infty} a_n z^n + z^{\sigma_2} \sum_{n=0}^{\infty} b_n z^n.$$

### The point at infinity

We set  $\zeta = \frac{1}{z}$  and examine what happens as  $\zeta \rightarrow 0$ . (3.1) becomes

$$\frac{d^2 w}{d\zeta^2} + \left( \frac{2}{\zeta} - \frac{p}{\zeta^2} \right) \frac{dw}{d\zeta} + \frac{1}{\zeta^4} q w = 0. \quad (3.5)$$

We then apply all our previous results to equation (3.5). We find that the point at infinity is a regular singular point if  $2 - zp(z)$  and  $z^2 q(z)$  are regular at infinity.

### 3.1.1 Bessel's Equation

We apply our theory to Bessel's equation

$$w'' + \frac{1}{z} w' + \left( 1 - \frac{\nu^2}{z^2} \right) w = 0. \quad (3.6)$$

This arises frequently in cylindrical geometries.  $\nu$  is a constant parameter. (3.6) has a regular singular point at  $z = 0$  and an irregular singular point at infinity. The indicial equation (at  $z = 0$ ) is  $\sigma^2 = \nu^2$ . We look for Frobenius series solutions of the form

$$w = z^\sigma \sum_{n=0}^{\infty} a_n z^n, \quad a_0 \neq 0.$$

We get a recurrence for  $a_n$ ,  $a_n n(n+2\sigma) = -a_{n-2}$ . We thus split our study of the form of the solutions of (3.6) according to  $\nu$ .

**Case 1.**  $2\nu \notin \mathbb{N}$ . Then  $\sigma_1 - \sigma_2 = 2\nu \notin \mathbb{N}$  and we get two series solutions. The coefficients of the equation can be determined by the recurrence and, on choosing  $a_0$  appropriately, we get the standard Bessel function  $J_\nu(z)$ .

$$J_\nu(z) = \left( \frac{1}{2} z \right)^\nu \sum_{k=0}^{\infty} \frac{\left( -\frac{1}{4} z^2 \right)^k}{k! \Gamma(k + \nu + 1)} \quad (3.7)$$

The series is clearly entire, the only finite singularity being the branch point at  $z = 0$  from the  $z^\nu$  factor. We get the linearly independent solution  $J_{-\nu}$  by replacing  $\nu$  with  $-\nu$  in (3.7).

**Case 2.**  $\nu = 0$ . Then  $\sigma_1 = \sigma_2 = 0$ . We get the solution  $J_0(z)$  ( $\nu = 0$  in (3.7)) and a second solution with a logarithmic singularity at  $z = 0$ ,

$$w_2(z) = J_0(z) \log z + \sum_{n=0}^{\infty} b_n z^n.$$

**Case 3.**  $2\nu \in \mathbb{N}$ . This has two subcases: **Case 3a.**  $\nu \in \mathbb{N}$ . Then we get  $J_n$  as before and a second logarithmic solution. **Case 3b.**  $2\nu$  is odd. This is one of the “black magic” cases referred to earlier. The first solution with  $\sigma = \nu > 0$  works to give  $J_\nu$  (it always works). The recurrence with  $\sigma = -\nu$  could potentially go wrong, but it just jumps over the trouble and we get  $J_{-\nu}$ . If you wish to persuade yourself of this just take a specific case (say  $\nu = \frac{3}{2}$ ) and play with it. The integer plus a half Bessel functions  $J_{n+\frac{1}{2}}(z)$  are all expressible in terms of elementary functions, for instance  $J_{\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi z}} \sin z$  and  $J_{-\frac{1}{2}}(z) = \frac{2}{\sqrt{\pi z}} \cos z$ .

## 3.2 Classification of equations by singularities

We will only consider second order equations with at most three regular singular points (including at infinity). This class, although it seems restrictive, in fact covers most of the differential equations of mathematical physics.

It will be convenient to ensure that our singularities are in nice places. The Möbius transform is useful, and we write it as

$$z \mapsto \frac{(z - \alpha)(\beta - \gamma)}{(\beta - \alpha)(z - \gamma)}. \quad (3.8)$$

This maps  $\alpha \mapsto 0$ ,  $\beta \mapsto 1$  and  $\gamma \mapsto \infty$ , and this seems a good place to point out that our discussion will be on the complex sphere  $\mathbb{C}_\infty$  (or the complex plane with the point at infinity attached).

### 3.2.1 Equations with no regular singular points

This will be a rather brief discussion.

**Proposition.** *There are no second order linear differential equations with no regular singular points.*

*Proof.* Since there are no finite singularities  $p$  and  $q$  are entire. We must have  $p \sim \frac{2}{z}$  as  $z \rightarrow \infty$ , so  $p$  is bounded and thus constant. We now have a contradiction.  $\square$

### 3.2.2 Equations with one regular singular point

Without loss of generality we can assume that this point is at  $z = 0$ . Thus  $p = \frac{A(z)}{z}$  and  $q = \frac{B(z)}{z^2}$ , with  $A(z)$  and  $B(z)$  entire functions. As the equation is regular at  $\infty$  we must have  $A = 2$  and  $B = 0$ . Thus the only second order linear differential equation with one regular singular point is

$$w'' + \frac{2}{z}w' = 0. \quad (3.9)$$

This has a general solution  $w(z) = \frac{\alpha}{z} + \beta$ .

### 3.2.3 Equations with two regular singular points

WLOG we can put these singular points at 0 and  $\infty$ , so we must have  $p = \frac{A(z)}{z}$  and  $q = \frac{B(z)}{z^2}$  with  $A$  and  $B$  entire. As  $z \rightarrow \infty$  we must have  $zp(z)$  and  $z^2q(z)$  bounded and thus  $A$  and  $B$  are bounded entire functions and therefore constant.

Thus the most general equation with two regular singular points at 0 and  $\infty$  is

$$z^2w'' + Azw' + Bw = 0. \quad (3.10)$$

This is a homogenous equation and so

$$w(z) = \begin{cases} \alpha z^{\sigma_1} + \beta z^{\sigma_2} & \sigma_1 \neq \sigma_2 \\ z^\sigma (\alpha + \beta \log z) & \sigma_1 = \sigma_2 = \sigma. \end{cases}$$

We can work backwards to find  $A$  and  $B$  in terms of  $\sigma_1$  and  $\sigma_2$ ; we find  $A = 1 - \sigma_1 - \sigma_2$  and  $B = \sigma_1\sigma_2$ .

#### Confluence of singularities

We map  $z \mapsto z + \alpha$  and let  $\alpha \rightarrow \infty$  and define  $\lambda_i = \alpha\sigma_i$ . Then we get the equation

$$w'' + (\lambda_1 + \lambda_2)w' + \lambda_1\lambda_2w = 0, \quad (3.11)$$

which has the general solution  $w(z) = \alpha e^{\lambda_1 z} + \beta e^{\lambda_2 z}$ . (3.11) has an irregular singular point at infinity and the solution has an essential singularity there.

### 3.2.4 Equations with three regular singular points

Or, a User's Guide to the hypergeometric equation. This section is only vaguely on the edge of the Schedules. We can assume that the singular points are at 0, 1 and  $\infty$ , and let the indices at 0 be 0,  $1 - c$ , at 1 be 0,  $c - a - b$  and at infinity be  $a, b$  respectively. We get the hypergeometric equation

$$z(z-1)w'' + [(a+b-1)z+c]w' + abw = 0. \quad (3.12)$$

There is one solution which is regular at the origin. It is written  $F(a, b, c; z)$  and

$$F(a, b, c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots \quad (3.13)$$

This series is convergent for  $|z| < 1$ . The other solution is  $z^{1-c}F(1+a-c, 1+b-c, 2-c; z)$  if  $c \notin \mathbb{Z}$ . If  $c \in \mathbb{Z}$  the second solution is logarithmic.

### Transformations to hypergeometric form

Suppose we have a second order linear differential equation of the form  $w'' + pw' + qw = 0$  with three regular singular points at  $z = A$ ,  $z = B$  and  $z = C$ . We transform this into hypergeometric form by applying a Möbius transform on the independent variable taking  $(A, B, C) \mapsto (0, 1, \infty)$  to put the singularities in the right place and applying a transform of the form  $w_{\text{old}} = (z - A)^\xi (z - B)^\eta (z - C)^\zeta w_{\text{new}}$  on the dependent variable to give the correct indices at the singularities. We illustrate this with an example, Legendre's equation:

$$(1 - z^2)w'' - 2zw' + \left(n(n+1) - \frac{m^2}{1 - z^2}\right) = 0, \quad (3.14)$$

which has regular singularities at  $\pm 1$  and  $\infty$ . The indices at  $\pm 1$  are  $\pm \frac{m}{2}$  and the indices at infinity are  $-n$  and  $n + 1$ . The transform

$$w(z) = (1 - z)^{\frac{m}{2}} (1 + z)^{\frac{m}{2}} f(z)$$

gives  $f$  indices of  $0$ ,  $-m$  at  $z = \pm 1$  and  $m - n, m + n + 1$  at  $z = \infty$ . and we also put  $\zeta = \frac{1-z}{2}$  to move the singularities at  $\pm 1$  to  $0$  and  $1$  respectively. The coefficients  $a$ ,  $b$  and  $c$  of the hypergeometric equation are therefore  $a = m - n$ ,  $b = m + n + 1$  and  $c = m + 1$  and so

$$w(z) = (1 - z^2)^{\frac{m}{2}} F(m - n, m + n + 1, m + 1; \frac{1-z}{2})$$

is a solution of (3.14).

### Integral representation

The point of departure is the series for  $F(a, b, c; z)$ , (3.13). We have

$$\begin{aligned} F(a, b, c; z) &= \sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{\Gamma(a)} \frac{\Gamma(k+b)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(k+c)} \frac{z^k}{k!} \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{\Gamma(a)} \frac{\Gamma(k+b)}{\Gamma(k+c)} \frac{\Gamma(c-b)}{k!} z^k \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{\Gamma(k+a)}{\Gamma(a)} B(b+k, c-b) \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\Gamma(k+a)}{\Gamma(a)} \int_0^1 t^{b+k-1} (1-t)^{c-b-1} dt \\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt. \end{aligned}$$

We thus get the final result

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} dt. \quad (3.15)$$

**Confluent hypergeometric equation**

We move the singularity at  $z = 1$  to  $z = b$  using  $z \mapsto bz$  and then let  $b \rightarrow \infty$ . We get the confluent hypergeometric equation

$$zw'' + (c - z)w' - aw = 0. \quad (3.16)$$

This has a regular singular point at  $z = 0$  with indices 0 and  $1 - c$  and an irregular singular point at infinity. The regular solution is

$$\Phi(a, c; z) = 1 + \frac{a}{c}z + \frac{a(a+1)}{c(c+1)}\frac{z^2}{2!} + \dots \quad (3.17)$$

and the other solution is  $z^{1-c}\Phi(1+a-c, 2-c; z)$  if  $c \notin \mathbb{Z}$ . The series representation (3.17) is entire.

If  $a = c$  we get  $\Phi(a, a; z) = e^z$ . If  $-a \in \mathbb{N}$  the series terminates to give the Laguerre polynomials. Hermite's equation

$$w'' - 2zw' + 2nw = 0 \quad (3.18)$$

has solutions  $\Phi(-\frac{1}{2}n, -\frac{1}{2}; z^2)$  and  $z^{\frac{1}{2}}\Phi(\frac{1-n}{2}, -\frac{3}{2}; z^2)$ . After some work we can get the Bessel functions

$$J_\nu(z) \propto z^\nu e^{-iz} \Phi(\nu + \frac{1}{2}, 2\nu + 1; 2iz). \quad (3.19)$$

**Triple confluence**

This can be done in a symmetric way by placing the singularities at  $K$ ,  $Ke^{\frac{2\pi i}{3}}$  and  $Ke^{\frac{4\pi i}{3}}$  with indices  $\frac{1}{6} \pm \frac{1}{3}K^{\frac{2}{3}}$  and letting  $K \rightarrow \infty$ . This results in Airy's equation

$$w'' - zw = 0, \quad (3.20)$$

which has no singularities in the finite complex plane but a really evil singularity at infinity.

If we let  $\zeta = \frac{2}{3}z^{\frac{3}{2}}$  and define  $W(\zeta) = z^{-\frac{1}{2}}w(z)$  we get

$$W'' + \frac{1}{3}W' + \left(1 - \frac{1}{9\zeta^2}\right)W = 0.$$

This is Bessel's equation (3.6) for  $W(i\zeta)$ , so we get  $w(z) = z^{\frac{1}{2}}J_{\pm\frac{1}{3}}\left(\frac{2i}{3}z^{\frac{3}{2}}\right)$  as a solution of (3.20).

**3.3 Integral representation of solutions**

In general, look for a solution of the form

$$w(z) = \int_{\Gamma} K(z, t)f(t) dt, \quad (3.21)$$

where we have freedom to choose  $K$ ,  $f$  and  $\Gamma$  so as to satisfy the differential equations.  $K(z, t)$  is known as the *kernel*. Some (famous?) kernels are:

1. Laplace kernel:  $K(z, t) = e^{zt}$ . This is used in Laplace transforms in the form  $e^{-zt}$  with  $\Gamma = \mathbb{R}^+$ .
2. Fourier kernel:  $K(z, t) = e^{izt}$ . This is used in Fourier transforms with  $\Gamma = \mathbb{R}$ .
3. Euler kernel:  $(t - z)^\mu$ .
4. Mellin kernel:  $t^{-z}$ .

The Laplace kernel and Fourier kernel amount to the same thing, the choice between them just influences  $\Gamma$ . We will only examine this kernel. Use is best illustrated by example.

Consider Airy's equation (3.20), and look for a solution of the form

$$w(z) = \int_{\Gamma} e^{zt} f(t) dt,$$

where  $\Gamma$  and  $f$  are to be determined. We substitute into (3.20) to get

$$\begin{aligned} 0 &= \int_{\Gamma} (t^2 - z)e^{zt} f(t) dt \\ &= \int_{\Gamma} (t^2 f(t) + f'(t)) e^{zt} dt - [e^{zt} f(t)]_{\Gamma} \quad \text{integrating by parts.} \end{aligned}$$

So if by hook or by crook we can find  $f$  such that the integrand is zero and  $[e^{zt} f(t)]_{\Gamma} = 0$  we have found a solution to (3.20). We choose  $f$  such that  $f' + t^2 f = 0$ , which gives  $f(t) = Ae^{-\frac{t^3}{3}}$  and

$$w(z) = \int_{\Gamma} e^{zt} e^{-\frac{t^3}{3}} dt. \quad (3.22)$$

We now have to choose  $\Gamma$  such that  $[e^{zt} e^{-\frac{t^3}{3}}]_{\Gamma} = 0$ . As we are dealing with an analytic function this is true on any closed  $\Gamma$ , but in this case (3.22) gives the true but not-very-useful solution  $w(z) \equiv 0$  (by Cauchy's theorem).

We can get this if we integrate over an infinite range and the integrand tends to zero at infinity. This happens iff  $-\frac{\pi}{2} < \arg t^3 < \frac{\pi}{2}$ , which is in the shaded regions of the  $t$ -plane.

Contours starting and ending in the same sector give  $w \equiv 0$ , so we have three choices of contour,  $\Gamma_1$ ,  $\Gamma_2$  and  $\Gamma_3$ . We note that

$$\int_{\Gamma_3} = \int_{\Gamma_1} - \int_{\Gamma_2}$$

and so we only have two linearly independent solutions. One choice is

$$w_{1,2} = \int_{\Gamma_{1,2}} e^{zt} e^{-\frac{t^3}{3}} dt.$$

Although this seems on the face of it to not be overly helpful we will see later that this can be approximately evaluated when  $|z| \gg 1$ , which is usually the physical case we are interested in.

For another example we try the confluent hypergeometric equation (3.16). We try as before

$$w(z) = \int_{\Gamma} e^{zt} f(t) dt$$

and find that this works if

$$[e^{zt}(t^2 - t)f(t)]_{\Gamma} - \int_{\Gamma} \left( \frac{d}{dt} \{ (t^2 - t)f \} - ctf + af \right) e^{zt} dt = 0.$$

As before we choose  $f(t)$  to make the integrand zero, which gives  $f(t) = (t - 1)^{c-a-1} t^{a-1}$  and then choose  $\Gamma$  to make  $[e^{zt} t^a (1 - t)^{c-a}]_{\Gamma} = 0$ . Choosing  $\Gamma$  depends on the particular ranges of  $a$ ,  $c$  and  $a - c$ , and given a range of  $a$ ,  $c$  and  $a - c$  it is not difficult to find  $\Gamma$  such that  $[e^{zt} t^a (1 - t)^{c-a}]_{\Gamma} = 0$  and the integral for  $w(z)$  does not give the trivial zero solution.



## Chapter 4

# Asymptotic Expansions

### 4.1 Motivation

We will motivate this discussion with an example. Suppose we wish to evaluate the error function

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} ds. \quad (4.1)$$

This occurs throughout statistics and mathematical physics, for instance as a solution to the diffusion equation  $\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$  (put  $z = \frac{x}{2\sqrt{\kappa t}}$  to get an ODE for  $T(z)$ ). One naive approach is to expand  $e^{-s^2}$  as an infinite sum and integrate termwise, which is certainly analytically permissible. As  $e^{-s^2}$  is entire then the series we obtain for  $\operatorname{erf} z$  will have an infinite radius of convergence. The series is

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} z \left( 1 - \frac{1}{3}z^2 + \frac{1}{10}z^4 - \frac{1}{42}z^6 + \dots \right). \quad (4.2)$$

If we evaluate this at  $z = 1$  we need eight terms to get an accuracy of  $10^{-5}$ . If we evaluate at  $z = 2$  we need 16 terms and if we evaluate at  $z = 5$  we need 75 terms. Although the series *is* convergent, the terms of the series get quite large before eventually tending to zero. At  $z = 5$  the largest term is approximately  $7 \times 10^8$ , so although a computer (say) can perform the sum of 75 terms in no time at all it will converge to something which is incorrect even in the first significant digit.

For large  $|z|$  a better approach is to obtain an asymptotic expansion for  $\operatorname{erf} z$ . We know that  $\operatorname{erf} z \rightarrow 1$  as  $z \rightarrow \infty$  and so we write

$$\begin{aligned} \operatorname{erf} z &= 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2} ds \\ &= 1 - \frac{2}{\sqrt{\pi}} \int_z^\infty s e^{-s^2} \frac{1}{s} ds \\ &= 1 - \frac{1}{\sqrt{\pi}} \frac{e^{-z^2}}{z} + \frac{1}{\sqrt{\pi}} \int_z^\infty \frac{e^{-s^2}}{s^2} ds \quad \text{integrating by parts.} \end{aligned}$$

We can continue this to get the asymptotic series for  $\operatorname{erf} z$

$$\operatorname{erf} z = 1 - \frac{e^{-z^2}}{z\sqrt{\pi}} \left( 1 - \frac{1}{2z^2} + \frac{1 \times 3}{(2z^2)^2} - \frac{1 \times 3 \times 5}{(2z^2)^3} + \mathcal{R} \right). \quad (4.3)$$

If we apply the ratio test we see that this series is convergent nowhere. However, if we consider the remainder term  $\mathcal{R}$  we see that

$$\begin{aligned}\mathcal{R} &= \int_z^\infty \frac{105}{16} \frac{te^{-t^2}}{t^9} dt \\ &\leq \frac{105}{32} \frac{e^{-z^2}}{z^9}\end{aligned}$$

and so the remainder term tends to zero very rapidly as  $z \rightarrow \infty$ . At  $z = 2.5$  only three terms of the series are needed for an accuracy of  $10^{-5}$  and at  $z = 3$  two terms will do. The truncated series is an asymptotic expansion of  $\operatorname{erf} z$  as  $z \rightarrow \infty$ . Note that the Taylor expansion is an asymptotic expansion of  $\operatorname{erf} z$  as  $z \rightarrow 0$ .

## 4.2 Definitions and examples

**Definition.** The sum  $\sum_{n=1}^N f_n(z)$  is an asymptotic expansion of  $f(z)$  in the limit  $z \rightarrow \infty$  if  $\forall M \leq N$  we have

$$\frac{f(z) - \sum_{n=1}^M f_n(z)}{f_M(z)} \rightarrow 0 \text{ as } z \rightarrow \infty.$$

We can state this informally as “the remainder is smaller than the last included term” and a similar definition holds in any limit  $z \rightarrow c$ . The property of asymptoticness may depend on  $\arg(z - c)$ .

**Definition.** A sequence of function  $\{\phi_n(z)\}_{n=0}^\infty$  is an asymptotic sequence as  $z \rightarrow c$  in some sector if  $\forall n, \frac{\phi_{n+1}(z)}{\phi_n(z)} \rightarrow 0$  as  $z \rightarrow c$  in that sector.

For instance  $\phi_n(z) = z^{-n}$  is asymptotic as  $z \rightarrow \infty$  with any argument.  $\phi_n(z) = e^{-nz}$  is asymptotic as  $z \rightarrow \infty$  for  $-\frac{\pi}{2} < \arg z < \frac{\pi}{2}$ .  $\phi_n(z) = (\sin z)^n$  is asymptotic as  $z \rightarrow 0$  for any argument.

**Definition.** If for a given asymptotic sequence  $\{\phi_n(z)\}$  there exist constants  $\{a_n\}$  such that for all  $n$ ,  $f(z) = \sum_0^N a_n \phi_n(z) = o(\phi_N(z))$  as  $z \rightarrow c$  in some sector then we write

$$f(z) \sim \sum_0^\infty a_n \phi_n(z) \text{ as } z \rightarrow c.$$

These infinite asymptotic expansions go against the spirit of their use, but they are conceptually useful as they allow us to show

$$f(z) = \sum_{n=0}^N a_n \phi_n(z) + a_{N+1} \phi_{N+1}(z) + o(\phi_{N+1}) = \sum_{n=0}^N a_n \phi_n(z) + \mathcal{O}(\phi_{N+1}).$$

For a given asymptotic sequence  $\{\phi_n\}$  the coefficients  $a_n$  are unique and can be found recursively from

$$a_N = \lim_{z \rightarrow c} \frac{f(z) - \sum_0^{N-1} a_n \phi_n}{\phi_N},$$

remembering that possibly  $z \rightarrow c$  in some sector. A given function can have different asymptotic expansions in terms of different asymptotic sequences:

$$\begin{aligned}\tan z &\sim z + \frac{1}{3}z^3 + \frac{2}{15}z^5 \\ &\sim \sin z + \frac{1}{2}(\sin z)^3 + \frac{3}{8}(\sin z)^5,\end{aligned}$$

both as  $z \rightarrow 0$  for any  $\arg z$ .

### 4.2.1 Manipulations

Asymptotic expansions can be naively added, subtracted, multiplied and divided to form new asymptotic expansions, but perhaps in terms of a new asymptotic series. The size of terms must be checked. Obviously, if we have an asymptotic expansion of  $f_1$  about  $c$  confined to a sector  $S_1$  and an asymptotic expansion of  $f_2$  about  $c$  confined to a sector  $S_2$  then the asymptotic expansion obtained by multiplication is only valid in the sector  $S_1 \cap S_2$ .

Asymptotic expansions can be integrated termwise but cannot in general be differentiated. However if  $f(z)$  is analytic in a sector and differentiable in the sector at  $c$  (some kind of one-sided limit) then the asymptotic expansion can be differentiated termwise in that sector.

## 4.3 Stokes Phenomenon

Suppose  $f(z) \sim \sum_N^\infty a_n z^{-n}$  as  $z \rightarrow \infty$  for all  $\arg z$ . Let  $f$  be analytic in a punctured neighbourhood of infinity. Then  $f$  has a (convergent) Laurent expansion  $\sum_{-\infty}^\infty b_n z^{-n}$ . This is asymptotic so by uniqueness  $b_n = 0$  for  $n < N$  and  $a_n = b_n$  for  $n \geq N$  and the asymptotic expansion is convergent.

Conversely if the asymptotic expansion is divergent then it cannot be valid for all  $\arg z$ . Divergent asymptotic expansions are associated with essential singularities of  $f$ . For instance we have seen that for real positive  $z$  we have  $\operatorname{erf} z \sim 1 - \frac{e^{-z^2}}{z\sqrt{\pi}}$  as  $z \rightarrow \infty$ . In deriving this we considered  $\int_z^\infty e^{-s^2} ds$ , and for more general  $\arg z$  this integral can be shifted onto the original contour provided we're in the sector such that  $e^{-s^2} \rightarrow 0$  as  $s \rightarrow \infty$ ; thus the asymptotic expansion is valid for  $-\frac{\pi}{4} < \arg z < \frac{\pi}{4}$ . Noting that  $\operatorname{erf}$  is an odd function of  $z$  we see that  $\operatorname{erf} z \sim -1 - \frac{e^{-z^2}}{z\sqrt{\pi}}$  as  $z \rightarrow \infty$  for  $\frac{3\pi}{4} < \arg z < \frac{5\pi}{4}$ .

An alternative method gives that  $\operatorname{erf} z \sim -\frac{e^{-z^2}}{z\sqrt{\pi}}$  as  $z \rightarrow \infty$  for  $\frac{\pi}{4} < \arg z < \frac{3\pi}{4}$  and  $\frac{5\pi}{4} < \arg z < \frac{7\pi}{4}$ . The lines separating these sectors are called Stokes lines; asymptotic expansions have jump discontinuities across Stokes lines.

## 4.4 Asymptotic Approximation of Integrals

### 4.4.1 Integration by parts

This is used if the independent variable is in a limit of the integral.

For instance, consider the exponential integral (with  $z$  real and positive at first)

$$\begin{aligned}
E_1(z) &= \int_z^\infty \frac{e^{-s}}{s} ds \\
&= \left[ -\frac{e^{-s}}{s} \right]_z^\infty - \int_z^\infty \frac{e^{-s}}{s^2} ds \\
&= \frac{e^{-z}}{z} \left( 1 - \frac{1}{z} \right) + 2 \int_z^\infty \frac{e^{-s}}{s^3} ds
\end{aligned}$$

We can bound the remainder term with  $\frac{2}{z^3}e^{-z} = o\left(\frac{e^{-z}}{z^2}\right)$  and so

$$E_1(z) \sim \frac{e^{-z}}{z} \left( 1 - \frac{1}{z} \right) \quad \text{as } z \rightarrow \infty \text{ for positive real } z.$$

For complex  $z$  we can see that the above method works if  $e^{-z} \rightarrow 0$  as  $z \rightarrow \infty$ , that is for  $\Re z > 0$ . The result is in fact true for all arguments of  $z$ , but we need another method to cope with that.

As another example we will find asymptotic expansions as  $z \rightarrow \infty$  for the Fresnel integrals (at first for positive real  $z$ )

$$c(z) = \int_0^z \cos t^2 dt \qquad s(z) = \int_0^z \sin t^2 dt.$$

We will consider

$$\begin{aligned}
f(z) &= \int_0^z e^{it^2} dt \\
&= \int_0^\infty e^{it^2} dt - \int_z^\infty e^{it^2} dt.
\end{aligned}$$

The first of these integrals can be done as a standard contour integral to give  $\frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}}$ . As for the second:

$$\begin{aligned}
\int_z^\infty e^{it^2} dt &= \int_z^\infty \frac{2ite^{it^2}}{2it} dt \\
&= \left[ \frac{e^{it^2}}{2it} \right]_z^\infty + \mathcal{O}(z^{-2}) \\
&= -\frac{e^{iz^2}}{2iz} + \mathcal{O}(z^{-2}).
\end{aligned}$$

The evaluation of this second integral carries over to negative real  $z$ . We need to change the first integral and we get

$$f(z) \sim \pm \frac{\sqrt{\pi}}{2}e^{i\frac{\pi}{4}} + \frac{e^{iz^2}}{2iz} \quad \text{as } z \rightarrow \pm\infty \in \mathbb{R}.$$

We ask if we can extend these results into more of  $\mathbb{C}$ . The key point is the  $e^{iz^2}$  term, which must decay as  $z \rightarrow \infty$ . This restricts the series to the regions  $0 \leq \arg z \leq \frac{\pi}{2}$  and  $\pi \leq \arg z \leq \frac{3\pi}{2}$  respectively.

In the quadrants with exponential growth the exponential term of the series dominates and we get  $f(z) \sim \frac{e^{iz^2}}{2iz}$  as  $z \rightarrow \infty$ . The real and imaginary axes are clearly Stokes lines.

#### 4.4.2 Watson's Lemma

This applies to integrals of the form

$$\int_0^A e^{-zt} g(t) dt \quad (4.4)$$

and relies on the fact that (in this case),  $e^{-zt}$  decays rapidly as  $|z| \rightarrow \infty$  with  $\Re z > 0$ . The integral is dominated by a neighbourhood of  $t = 0$ .

**Lemma (Watson's Lemma).** *Suppose  $g(t)$  has an asymptotic expansion in a sector  $S$ ,*

$$g(t) \sim a_0 t^{\alpha_0} + \dots + a_n t^{\alpha_n} \quad \text{as } t \rightarrow 0$$

with  $\alpha_0 > -1$ . Then the integral (4.4) can be evaluated termwise and

$$\begin{aligned} \int_0^A e^{-zt} g(t) dt &\sim \int_0^{\infty} e^{-zt} (a_0 t^{\alpha_0} + \dots + a_n t^{\alpha_n}) dt \\ &\sim \sum_0^n a_k z^{-\alpha_k - 1} \Gamma(\alpha_k + 1) \end{aligned}$$

as  $z \rightarrow \infty$  with  $z^{-1}$  and  $A$  in  $S$ .

We do not prove this but give examples of its use. Consider (again) the exponential integral

$$\begin{aligned} \int_z^\infty \frac{e^{-s}}{s} ds &= e^{-z} \int_0^\infty \frac{e^{-zt}}{t+1} dt \\ &\sim e^{-z} \int_0^\infty e^{-zt} (1 - t + t^2 - \dots) dt \\ &\sim e^{-z} \left(1 - \frac{1}{z}\right) \end{aligned}$$

as  $z \rightarrow \infty$ , and as the Taylor series for  $(1+t)^{-1}$  we used is asymptotic as  $t \rightarrow 0$  for any argument the end result we get for  $E_1$  is valid for any argument. Using similar artifice we can do the same sort of thing for the error function, although it is easier to work with the complementary error function  $\operatorname{erfc} z = 1 - \operatorname{erf} z$ .

#### 4.4.3 Laplace's Method

This applies to integrals of the form

$$I(x) = \int_\alpha^\beta g(t) e^{xh(t)} dt \quad (4.5)$$

in the asymptotic limit  $x \rightarrow \infty$  (which is understood to mean  $x \gg 1$ ). The integrand is largest where  $h$  has its maximum. If the maximum is at an endpoint (say at  $\alpha$ ) with  $h'(\alpha) < 0$  then by expanding  $h(t) = h(\alpha) + (t - \alpha)h'(\alpha) + \mathcal{O}(t - \alpha)^2$  and expanding  $g(t) = g(\alpha) + \mathcal{O}(t - \alpha)$  we get

$$I(x) \sim -\frac{e^{xh(\alpha)}g(\alpha)}{xh'(\alpha)}.$$

The remainder term is  $\mathcal{O}(x^{-2})$  and so this an asymptotic expansion. We get the leading order term easily but the higher order terms are unpleasant.

If there is an interior maximum (at  $t_0$ ) then only the highest maximum contributes to the leading order term. Then  $h(t) = h_0 + \frac{1}{2}h_2(t - t_0)^2 + \mathcal{O}(t - t_0)^3$  with  $h_2 < 0$ . Now

$$\begin{aligned} I &= e^{xh_0} \int_{\alpha}^{\beta} e^{x(\frac{1}{2}h_2(t-t_0)^2 + \mathcal{O}(t-t_0)^3)} g(t) dt \\ &\sim \int_{-\infty}^{\infty} e^{\frac{x}{2}h_2u^2} (1 + \mathcal{O}(u^4x)) (g_0 + \mathcal{O}(u^2)) du \\ &\sim g_0 e^{h_0x} \left\{ \frac{2\pi}{-h_2x} \right\}^{\frac{1}{2}} + g_0 e^{h_0x} \mathcal{O}(x^{-\frac{3}{2}}). \end{aligned}$$

Note that the  $\mathcal{O}(u^3x)$  and  $\mathcal{O}(ux)$  terms are lost (integrating an odd function).

This gives us an easy way to derive Stirling's formula for  $x!$ . Recall that  $x! = \int_0^{\infty} t^x e^{-t} dt$ . Put  $t = x\tau$  to get

$$\begin{aligned} x! &= \int_0^{\infty} e^{x(\log x + \log \tau)} e^{-x\tau} x d\tau \\ &= xx^x \int_0^{\infty} e^{x(\log \tau - \tau)} d\tau. \end{aligned}$$

The maximum of  $\log \tau - \tau$  is at  $\tau = 1$  and we apply the formula developed above to get  $x! \sim \sqrt{2\pi x} x^x e^{-x}$  as  $x \rightarrow \infty$ .

A harder example (which comes from scattering cross-sections in fusion reactions) is

$$I(a, b) = \int_0^{\infty} \exp \left\{ -\left(\frac{a}{t}\right)^{\frac{1}{2}} - \frac{t}{b} \right\} dt.$$

The integrand is peaked at  $\left(\frac{ab^2}{4}\right)^{\frac{1}{3}}$ . The integral is locally dominated by this region. We pick up the major contribution by rescaling  $t = (ab^2)^{\frac{1}{3}} \tau$ . We put  $x = \left(\frac{a}{b}\right)^{\frac{1}{3}}$  and let this tend to infinity. This gives

$$I(a, b) = a^{\frac{1}{3}} b^{\frac{2}{3}} \int_0^{\infty} e^{-x(\tau^{-\frac{1}{2}} + \tau)} d\tau$$

and is now in a suitable form for the application of Laplace's method. Applying this gives (eventually)

$$I(a, b) \sim \left(\frac{16\pi^3 ab^5}{27}\right)^{\frac{1}{6}} \exp - \left(\frac{27a}{4b}\right)^{\frac{1}{3}}.$$

#### 4.4.4 The method of stationary phase

We will need the Riemann-Lebesgue lemma, which is stated but not proved.

**Lemma (Riemann-Lebesgue).** *If  $f(t)$  is continuous in  $[a, b]$  then*

$$\int_a^b f(t) e^{ixt} dt \rightarrow 0 \text{ as } x \rightarrow \infty.$$

We can see intuitively that when  $x$  becomes large the exponential term is oscillating faster and faster and getting more and more cancellation.

The method of stationary phase applies to integrals of the form

$$f(x) = \int_a^b e^{ixh(t)} g(t) dt \quad x \gg 1,$$

where  $x$ ,  $g$  and  $h$  are all real.

First we note that if  $h$  is strictly monotonic in some subinterval  $[\alpha, \beta]$  then

$$\int_\alpha^\beta e^{ixh(t)} g(t) dt = \int_\alpha^\beta e^{ixh} \frac{g(t(h))}{h'(t(h))} dh$$

which tends to 0 as  $x \rightarrow \infty$  (by the Riemann-Lebesgue lemma). This change of variables also suggests that the dominant contributions to  $f$  are when  $h'(t) = 0$ . Near a stationary point  $t_0$  of  $h$  we expand  $h(t) = h_0 + \frac{1}{2}(t - t_0)^2 h_2 + \mathcal{O}(t - t_0)^3$  and then

$$\begin{aligned} f(x) &\sim \int_{-\infty}^{\infty} e^{ih_0 x} g(t_0) e^{ix\tau^2 h_2} d\tau \\ &\sim g(t_0) e^{ih_0 x} \int_{-\infty}^{\infty} \exp\left(\frac{1}{2} i h_2 x \tau^2\right) d\tau \\ &\sim \left[\frac{2\pi}{x h''(t_0)}\right]^{\frac{1}{2}} g(t_0) e^{ixh(t_0) + i\frac{\pi}{4}}. \end{aligned}$$

As an example we will consider the Airy function for large negative  $x$ .

$$Ai(-x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}\omega^3 - x\omega\right) d\omega.$$

This is most easily approached with

$$I(x) = \int_0^\infty e^{i\left(\frac{\omega^3}{3} - x\omega\right)} d\omega.$$

Let  $\omega = \sqrt{x}t$  and  $\lambda = x^{\frac{3}{2}}$ . We get

$$I(x) = \sqrt{x} \int_0^\infty e^{i\lambda\left(\frac{t^3}{3} - t\right)} dt,$$

which is now in the correct form for the method of stationary phase. Applying the theory gives

$$Ai(-x) \sim \left(\pi x^{\frac{3}{2}}\right)^{-\frac{1}{2}} \cos\left(\frac{2}{3}x^{\frac{3}{2}} - \frac{\pi}{4}\right) \text{ as } x \rightarrow \infty.$$

Another (more physical) example is that of group velocity. Many waves are dispersive — different wavelengths travel at different speeds. If the waves are linear they can be superposed and Fourier analysis used to obtain

$$f(x, t) = \int_{-\infty}^{\infty} F(k) e^{i(kx - \omega(k)t)} dk \quad (4.6)$$

given an initial disturbance

$$f(x, 0) = \int_{-\infty}^{\infty} F(k) e^{ikx} dk.$$

Suppose the initial disturbance is compact to  $|x| < a$ . What does one see at large distances from the initial disturbance? We also need  $t$  to be large, else no waves reach  $x$ . The method of stationary phase gives the asymptotic behaviour of (4.6) as dominated by the point where  $\frac{\partial}{\partial k}(kx - \omega(k)t) = 0$ , or in other words  $\frac{x}{t} = \frac{\partial \omega}{\partial k}$ . For a given large  $x, t$  find  $k_0$  such that  $\left. \frac{\partial \omega}{\partial k} \right|_{k_0} = \frac{x}{t}$ . Then the dominant waveform is  $F(k_0) e^{i(k_0 x - \omega(k_0)t)}$ .

Conversely, given  $k_0$  then an observer moving with speed  $\left. \frac{\partial \omega}{\partial k} \right|_{k_0}$  asymptotically sees waves of wavenumber  $k_0$ . This speed  $\left. \frac{\partial \omega}{\partial k} \right|_{k_0}$  is called the group velocity and is the physical quantity: the speed at which energy is transferred.

For water waves  $\omega = \sqrt{gk}$  and along a ray  $\frac{x}{t} = c_g = \frac{1}{2} \sqrt{\frac{g}{k}}$ . The method of stationary phase gives

$$f(x, t) \sim 2\sqrt{2\pi k_0}^{-\frac{3}{4}} t^{-\frac{1}{2}} g^{-\frac{1}{4}} F(k_0) e^{i(k_0 x - \omega(k_0)t + \frac{\pi}{4})}.$$

#### 4.4.5 Method of Steepest Descents

In this section we generalise the method of stationary phase (or equivalently Laplace's method) into the complex plane. We consider integrals of the form

$$f(z) = \int_{\gamma} g(\zeta) e^{zh(\zeta)} d\zeta$$

with  $h$  and  $g$  analytic on  $\gamma$ . To begin with we consider  $z$  real and positive and split  $h$  into real and imaginary parts,  $h = u + iv$ . We cannot naively apply the previous methods as if the maximum of  $u(z)$  on  $\gamma$  is not a saddle point of  $h$  then  $v$  is varying monotonically and in the limit a large amount of cancellation occurs.

We can deform  $\gamma$  into  $\gamma'$  to pass through a saddle point of  $h$  along a contour of  $v$ . In this case  $u$  has its maximum on  $\gamma'$  at the saddle point, so the integral is locally dominated *and*  $v$  is stationary so no rapid cancellation occurs.

By letting  $\gamma'$  be tangent to a contour of  $v$  at the saddle point we get  $u$  decreasing most rapidly either side of the saddle and  $e^{zh}$  is most strongly peaked. This is the method of steepest descents. If only the leading order term is required then any descending path through the saddle point will do — this is the saddle point method.

As an example we consider the Airy function

$$Ai(z) = \frac{1}{2\pi i} \int_{C_1} e^{zt - \frac{1}{3}t^3} dt$$

with  $z$  real and positive. We write  $t = z^{\frac{1}{2}}\tau$  and then

$$Ai(z) = \frac{z^{\frac{1}{2}}}{2\pi i} \int_{C_1} e^{z^{\frac{3}{2}}\left(\tau - \frac{\tau^3}{3}\right)} d\tau.$$

We now consider the integral

$$f(x) = \int_{C_1} e^{x\left(\tau - \frac{\tau^3}{3}\right)} d\tau$$

for  $x = z^{\frac{3}{2}} \gg 1$  and  $h(\tau) = \tau - \frac{\tau^3}{3}$ , which has saddles at  $\tau = \pm 1$ . Although  $\tau = 1$  is the higher saddle we will go through  $\tau = -1$  so that  $\Re h(\tau) < 0$  on the path of integration and  $Ai(z)$  remains bounded as  $z \rightarrow \infty$ .

Put  $\tau = -1 + \eta$  with  $\eta = \xi + i\zeta$ .  $h(\tau) = -\frac{2}{3} + \eta^2 + \dots$ . The steepest descent path at  $\tau = -1$  is parallel to the imaginary axis, so we have  $\xi = 0$ . Then

$$\begin{aligned} f(x) &\sim \int_{-\infty}^{\infty} e^{x\left(-\frac{2}{3} - \zeta^2\right)} i d\zeta \\ &\sim i e^{-\frac{2}{3}x} \int_{-\infty}^{\infty} e^{-x\zeta^2} d\zeta \\ &\sim i \sqrt{\frac{\pi}{x}} e^{-\frac{2}{3}x}. \end{aligned}$$

Putting all this together  $Ai(z) \sim \frac{z^{-\frac{1}{4}}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{\frac{3}{2}}}$ .

We also want to know how far we can generalise this result for different values of  $\arg z$ . Suppose  $\arg z = \alpha$ ; this causes rotation of the steepest descent paths of  $\Re(zh(\tau))$  by  $\alpha$ . For small  $\alpha$  this doesn't matter — we get the same asymptotic expansion. For larger  $\alpha$  other saddles come into view. As the steepest descent path jumps from one saddle to another we get Stokes' phenomenon. For the Airy function the asymptotic expansion we found is valid for  $|\arg z| < \pi$ .

The Hankel functions

$$H_\nu^{(1,2)} = \frac{1}{\pi i} \int_{C_{1,2}} e^{z \sinh t - \nu t} dt.$$

The Bessel function  $J_\nu(z) = \frac{1}{2} \left( H_\nu^{(1)}(z) + H_\nu^{(2)}(z) \right)$ . We will seek an asymptotic approximation for  $H_\nu^{(1)}(z)$ , and use the method of steepest descent with  $g(t) = e^{-\nu t}$  and  $h(t) = \sinh t$ . This has saddles where  $\cosh t = 0$ , or  $t = \left(n + \frac{1}{2}\right) i\pi$ . We deform  $C_1$  to go through the saddle at  $t_0 = \frac{i\pi}{2}$ .

$g(t_0) = e^{-i\frac{\pi\nu}{2}}$ ,  $h(t_0) = i$  and  $h'(t_0) = i$ . Then  $h \sim i + \frac{1}{2}(t - t_0)^2$  and put  $t - t_0 = re^{i\theta}$ . Then  $h \sim i - \frac{1}{2}r^2 e^{i(2\theta - \frac{\pi}{2})}$ . It is clear that the path of steepest descent has  $\theta = \frac{\pi}{4}$ . However, any  $\theta$  such that  $0 < \theta < \frac{\pi}{2}$ , since in this range  $\Re \left( e^{i(2\theta - \frac{\pi}{2})} \right) > 0$ . If we put  $2\alpha = 2\theta - \frac{\pi}{2}$  we have

$$\begin{aligned}
H_\nu^{(1)}(z) &\sim \frac{1}{\pi i} e^{-i\frac{\pi\nu}{2}} e^{iz} \int e^{-\frac{1}{2}zr^2} e^{2i\alpha} e^{i\theta} dr \\
&\sim \frac{1}{\pi i} e^{-i\frac{\pi\nu}{2}} e^{iz} \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} e^{-i\alpha} e^{i\theta} \\
&\sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i(z - \frac{\pi\nu}{2} - \frac{\pi}{4})}
\end{aligned}$$

## 4.5 Liouville-Green Functions

This area has a number of names associated with it; it is usually called WKB theory<sup>1</sup>.

We return to equations of the form  $w'' + p(z)w' + q(z)w = 0$  and by putting  $w(z) = W(z)e^{-\frac{1}{2}\int^z p(s)ds}$  we convert into the standard form  $w'' + q(z)w = 0$ .

If  $q$  is a constant then we can write down a solution of this equation;  $w(z) = Ae^{i\theta}$  where  $\theta = q^{\frac{1}{2}}z$ . When  $q > 0$  solutions are oscillatory with wavelength proportional to  $q^{-\frac{1}{2}}$ .

The WKBJ method can be applied to problems in which  $q$  varies slowly, that is  $\frac{\Delta q}{q}$  is small when  $\Delta z = \mathcal{O}(q^{-\frac{1}{2}})$ , or alternatively “the fractional change in  $q$  is small over one wavelength”.

This derivation is starred — the final result isn't.

Take  $\epsilon \ll 1$  and  $q = q(\epsilon z)$  so that  $\frac{dq}{dz} = \epsilon q' = \mathcal{O}(\epsilon)$ . We expect solutions with a slowly varying amplitude  $a = a(\epsilon z)$  and a slowly varying phase  $\theta = \epsilon^{-1}\phi(\epsilon z)$ . The factor  $\epsilon^{-1}$  ensures the wavelength is  $\mathcal{O}(1)$  to leading order. Then

$$\begin{aligned}
e^{i\theta} &\sim e^{i(\epsilon^{-1}\phi(0) + \phi'(0)z + \frac{1}{2}\epsilon\phi''(0)z^2 + \dots)} \\
&\sim e^{\frac{i\phi(0)}{\epsilon}} e^{i\phi'(0)z}.
\end{aligned}$$

We propose a solution  $w(z) = a(\epsilon z)e^{\frac{i\phi(\epsilon z)}{\epsilon}}$ . Substituting into the governing equation we get

$$\begin{aligned}
\mathcal{O}(1) : & \quad -a\phi'^2 + qa = 0 \\
\mathcal{O}(\epsilon) : & \quad 2a'\phi' + a\phi'' = 0.
\end{aligned}$$

The  $\mathcal{O}(1)$  equation gives  $\phi = \pm \int^z q^{\frac{1}{2}} dz$  and the  $\mathcal{O}(\epsilon)$  equation gives  $a = q^{-\frac{1}{4}}$  (integrating and using the  $\mathcal{O}(1)$  equation). We have the Liouville-Green approximate solutions to the differential equation:

$$w \sim q^{-\frac{1}{4}} \left\{ A \exp\left(i \int^z q^{\frac{1}{2}} dz\right) + B \exp\left(-i \int^z q^{\frac{1}{2}} dz\right) \right\}. \quad (4.7)$$

If  $q < 0$  a more convenient form is

$$w \sim (-q)^{-\frac{1}{4}} \left\{ a \exp\left(\int^z (-q)^{\frac{1}{2}} dz\right) + b \exp\left(-\int^z (-q)^{\frac{1}{2}} dz\right) \right\}. \quad (4.8)$$

<sup>1</sup>and in Cambridge, WKBJ theory. W, K, B and J are Wentzel, Kramers, Brillouin and Jeffrey respectively.

These give asymptotic solutions for large  $z$  under certain conditions:

Suppose  $q \sim z^n$  as  $z \rightarrow \infty$ . Then  $a \sim z^{-\frac{n}{4}}$  and  $\phi \sim z^{\frac{n}{2}+1}$ . Recalling the derivation we see that we can neglect the term corresponding to the  $\epsilon^2$  term provided  $a'' \ll a'\phi'$  as  $z \rightarrow \infty$ , or substituting in we get  $n > -2$ . This is the converse of the condition for the point at  $\infty$  to be a regular singular point. The WKBJ method thus works when the point at infinity is an irregular singular point.

As an example we consider Airy's equation  $w'' - zw = 0$  (again). Thus  $q = -z$  and  $q^{\frac{1}{2}} = \pm i z^{\frac{1}{2}}$ .  $\int \sqrt{q} dz = \pm i \frac{2}{3} z^{\frac{3}{2}}$  and so

$$\begin{aligned} w(z) &\sim z^{-\frac{1}{4}} \exp\left(\pm z^{\frac{3}{2}}\right) & z \rightarrow +\infty \\ &\sim |z|^{\frac{1}{4}} \begin{cases} \cos\left(\frac{2}{3}|z|^{\frac{3}{2}}\right) \\ \sin\left(\frac{2}{3}|z|^{\frac{3}{2}}\right) \end{cases} & z \rightarrow -\infty. \end{aligned}$$

We do not get the constants from this method.

### 4.5.1 Connection formulae

These Liouville-Green functions (4.7) and (4.8) work well where  $q > 0$  or  $q < 0$  but do not work where  $q$  passes through zero, at which points the frequency is zero ( $q$  is no longer slowly varying on scales of the wavelength, which becomes infinite) and the amplitude is infinite.

Points at which  $q = 0$  are called turning points and the equation. WLOG consider  $q < 0$  for  $z < 0$  and  $q > 0$  for  $z > 0$

In regions 1 and 3 we have the Liouville-Green solutions

$$\begin{aligned} w(z) &\sim \frac{1}{(-q)^{\frac{1}{4}}} [Ae^\phi + Be^{-\phi}] & \phi(z) &= \int^z (-q)^{\frac{1}{2}} dz \\ w(z) &\sim \frac{1}{q^{\frac{1}{4}}} [a \cos \theta + b \sin \theta] & \theta(z) &= \int^z q^{\frac{1}{2}} dz. \end{aligned}$$

We have four unknown constants. We get two equations from the boundary conditions at  $\pm\infty$  and two others from the connection formulae across region 2.

As  $z \rightarrow 0$ ,  $q(z) \sim q_1 z$  (since  $q(0) = 0$ ) and the above expansions become

$$w \sim \frac{1}{(-q_1 z)^{\frac{1}{4}}} [Ae^\phi + Be^{-\phi}] \quad \phi(z) = \frac{2}{3} q_1^{\frac{1}{2}} (-z)^{\frac{3}{2}} \quad (4.9)$$

$$w \sim \frac{1}{(q_1 z)^{\frac{1}{4}}} [a \cos \theta + b \sin \theta] \quad \theta(z) = \frac{2}{3} q_1^{\frac{1}{2}} z^{\frac{3}{2}}, \quad (4.10)$$

valid for  $z \rightarrow 0_-$  in region 1 and  $z \rightarrow 0_+$  in region 3 respectively. They need to be matched across the intermediate region 2. In this region we approximate the differential equation as  $w'' + q_1 z w \sim 0$  and on letting  $\tau = -(q_1)^{\frac{1}{3}} z$  we get Airy's

equation  $w'' - \tau w \sim 0$ . This has solutions  $w \sim \alpha Ai(\tau) + \beta Bi(\tau)$ . We can use steepest descents (for instance) to find asymptotic expressions for this inner solution as  $z \rightarrow \pm\infty$ . As  $z \rightarrow -\infty$ ,  $\tau \rightarrow +\infty$  and we get the asymptotic expression

$$w \sim \frac{1}{\sqrt{\pi} q_1^{\frac{1}{12}} (-z)^{\frac{1}{4}}} \left\{ \frac{1}{2} \alpha e^\phi + \beta e^{-\phi} \right\}. \quad (4.11)$$

As  $z \rightarrow \infty$  we have  $\tau \rightarrow -\infty$  and

$$w \sim \frac{1}{\sqrt{2\pi} q_1^{\frac{1}{12}} z^{\frac{1}{4}}} \{(\alpha - \beta) \sin \theta + (\alpha + \beta) \cos \theta\}. \quad (4.12)$$

We now need to match coefficients between (4.11) and (4.9); we get

$$\alpha = \frac{2\sqrt{\pi}}{q_1^{\frac{1}{6}}} A$$

$$\beta = \frac{\sqrt{\pi}}{q_1^{\frac{1}{6}}} B.$$

Doing the same thing with (4.12) and (4.10) we get

$$a = \frac{q_1^{\frac{1}{6}}}{\sqrt{2\pi}} (\alpha + \beta)$$

$$b = \frac{q_1^{\frac{1}{6}}}{\sqrt{2\pi}} (\alpha - \beta).$$

Eliminating  $\alpha$  and  $\beta$  we obtain the connection formulae

$$A = \frac{a + b}{2\sqrt{2}}$$

$$B = \frac{a - b}{\sqrt{2}}. \quad (4.13)$$

As an example of the use of the connection formulae we seek to find approximate energy eigenvalues for the non-dimensional Schrödinger equation  $\psi'' + (E - z^2) \psi = 0$  with the boundary conditions  $\psi \rightarrow 0$  as  $z \rightarrow \pm\infty$  (quantum harmonic oscillator). In particular we will consider  $E \gg 1$ . We see that there are oscillations of frequency  $E^{\frac{1}{2}}$  (and so the wavelength is proportional to  $E^{-\frac{1}{2}}$ ).  $q = E - z^2$  is varying on a scale of  $E^{\frac{1}{2}}$  and so is slowly varying on the scale of the oscillations. We can therefore apply WKBJ theory and the connection formulae.

There are turning points at  $\pm E^{\frac{1}{2}}$  and exponentially decaying solutions in  $|z| > E^{\frac{1}{2}}$ . In  $z < -E^{\frac{1}{2}}$  we have  $B = 0$  and WLOG  $A = 1$ . The connection formulae at  $z = -E^{\frac{1}{2}}$  give  $a = b = \sqrt{2}$  and so in  $|z| < E^{\frac{1}{2}}$  we have

$$\psi \sim \frac{1}{(E - z^2)^{\frac{1}{4}}} \left( \sqrt{2} \sin \theta + \sqrt{2} \cos \theta \right)$$

$$\sim \frac{2}{(E - z^2)^{\frac{1}{4}}} \sin \left( \theta + \frac{\pi}{4} \right)$$

where  $\theta = \int_{-\sqrt{E}}^z (E - z^2)^{\frac{1}{2}} dz$  (the lower limit puts the origin at 0 for the connection formulae). Near  $z = \sqrt{E}$  we write

$$\begin{aligned} \psi &\sim \frac{2}{q^{\frac{1}{4}}} \sin \left( \int_{-\sqrt{E}}^{\sqrt{E}} q^{\frac{1}{2}} dz - \int_z^{\sqrt{E}} q^{\frac{1}{2}} dz + \frac{\pi}{4} \right) \\ &\sim \frac{2}{q^{\frac{1}{4}}} \{ \sin \alpha \cos \theta' + \cos \alpha \sin \theta' \}, \end{aligned}$$

where  $\alpha = \int_{-\sqrt{E}}^{\sqrt{E}} q^{\frac{1}{2}} dz + \frac{\pi}{4}$  and  $\theta' = \int_z^{\sqrt{E}} q^{\frac{1}{2}} dz$ . This moves the origin to  $z = \sqrt{E}$  and we can now apply the connection formulae (4.13).

In  $z > \sqrt{E}$  we have  $A' = 0$  (for exponential decay) and so  $\psi = B'e^{-\phi}$ . Therefore we obtain

$$2 \cos \alpha = \frac{B'}{\sqrt{2}} \quad 2 \sin \alpha = -\frac{B'}{\sqrt{2}}.$$

This implies that  $\tan \alpha = -1$ , or that  $\alpha = n\pi - \frac{\pi}{4}$ . We can easily do the integral for  $\alpha$ , and we obtain  $E_n = n - \frac{1}{2}$  for  $n \in \mathbb{N}$ . We have (coincidentally) obtained the exact eigenvalues, and it is clear that this method can be used to find approximate eigenvalues of more complicated potentials.



# Chapter 5

## Laplace Transforms

### 5.1 Definition and simple properties

The Laplace transform of a function  $f(t)$  is defined by

$$F(p) = \mathcal{L}[f(t)] = \int_0^{\infty} e^{-pt} f(t) dt. \quad (5.1)$$

The variable  $p$  may be complex but we must have  $\Re(p) > \gamma$  where  $\gamma$  is sufficiently large to permit convergence. A greater class of functions have Laplace transforms than have Fourier transforms, due to the exponential attenuation at large  $t$ .

Since the integral's range is  $[0, \infty)$  we lose all knowledge of the function for  $t < 0$  and the inversion of  $\mathcal{L}[f(t)]$  is  $H(t)f(t)$ .

The following properties are both trivial to prove and very useful in both evaluating and inverting Laplace transforms.

- $\mathcal{L}[\lambda f + \mu g] = \lambda \mathcal{L}[f] + \mu \mathcal{L}[g]$
- shifting:  $\mathcal{L}[e^{at} f(t)] = F(p - a)$
- $\mathcal{L}[H(t - a)f(t - a)] = e^{-ap} F(p)$
- change of scale:  $\mathcal{L}[f(\alpha t)] = \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right)$
- $\mathcal{L}\left[\frac{df}{dt}\right] = -f(0) + pF(p)$
- $\mathcal{L}\left[\int_0^t f(u) du\right] = \frac{F(p)}{p}$
- $\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{dp^n} F(p)$

These properties often give the best way to calculate Laplace transforms and to guess inverses. We can now, starting from  $\mathcal{L}[1] = \frac{1}{p}$  obtain Laplace transforms for a reasonably useful class of functions.

$f$	$\mathcal{L}[F]$
$H(t - \alpha)$	$\frac{e^{-\alpha p}}{p}$
$t^n$	$\frac{n!}{p^{n+1}}$
$e^{\alpha t}$	$\frac{1}{p - \alpha}$
$\cos \alpha t$	$\frac{p}{p^2 + \alpha^2}$
$\sin \alpha t$	$\frac{\alpha}{p^2 + \alpha^2}$
$\cosh \alpha t$	$\frac{p}{p^2 - \alpha^2}$
$\sinh \alpha t$	$\frac{\alpha}{p^2 - \alpha^2}$

Table 5.1: Simple Laplace transforms

### 5.1.1 Asymptotic Limits

Using Watson's Lemma as  $p \rightarrow \infty$  we get

$$F(p) \sim \sum_{n=0}^{\infty} \frac{1}{p^{n+1}} f^{(n)}(0)$$

and so  $\lim_{p \rightarrow \infty} pF(p) = f(0)$ .

From the properties of the Laplace transform on the preceding page we have that

$$pF(p) = f(0) + \int_0^{\infty} e^{-pt} \frac{df}{dt} dt$$

and so letting  $p \rightarrow 0$  we get  $\lim_{p \rightarrow 0} pF(p) = \lim_{t \rightarrow \infty} f(t)$  (if both limits exist).

This begs the obvious question; how do we know that both limits exist? If all the singularities of  $F(p)$  lie in  $\{z \in \mathbb{C} : \Re z < 0\}$  then both the limits exist.

### 5.1.2 Convolutions

We define

$$f * g = \int_{-\infty}^{\infty} f(\tau)g(t - \tau) d\tau \quad (5.2)$$

and since  $f(y) = g(y) = 0$  for  $y < 0$  we have that

$$f * g = \int_0^t f(\tau)g(t - \tau) d\tau.$$

Now

$$\begin{aligned} \mathcal{L}[f * g] &= \int_0^{\infty} \int_0^t e^{-pt} f(\tau)g(t - \tau) d\tau dt \\ &= \int_0^{\infty} \int_{\tau}^{\infty} e^{-pt} f(\tau)g(t - \tau) dt d\tau \\ &= \int_0^{\infty} e^{-p\tau} f(\tau) d\tau \int_0^{\infty} e^{-pu} g(u) du \\ &= \mathcal{L}[f] \mathcal{L}[g]. \end{aligned}$$

## 5.2 Inversion

Consider

$$I = \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} F(p) \, dp$$

where the contour lies to the right of all of the singularities of  $F(p)$ .

At this stage you should be scenting waffle; recall that the Laplace transform was defined only for  $p$  with sufficiently large real part. What we mean by  $F(p)$  is the analytic continuation of the Laplace transform into the whole of  $\mathbb{C}$ .

We now evaluate  $I$  (recall that we insisted that  $f(t) = 0$  for  $t < 0$ ).

$$\begin{aligned} I &= \int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt} \int_0^\infty f(\tau) e^{-p\tau} \, d\tau \, dp \\ &= \int_{\gamma-i\infty}^{\gamma+i\infty} \int_{-\infty}^\infty e^{p(t-\tau)} f(\tau) \, d\tau \, dp \\ &= ie^{\gamma t} \int_{y=-\infty}^\infty \int_{\tau=-\infty}^\infty e^{iy(t-\tau)} f(\tau) \, d\tau \, dy \\ &= 2\pi i e^{\gamma t} \int_{-\infty}^\infty \delta(t-\tau) e^{-\gamma\tau} f(\tau) \, d\tau \\ &= 2\pi i f(t). \end{aligned}$$

We thus obtain the Bromwich inversion formula:

$$f(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} F(p) e^{pt} \, dp. \quad (5.3)$$

Since  $\gamma$  is chosen so that the contour of integration lies to the right of all the singularities we can close the contour in the right half-plane for  $t < 0$  and use Cauchy's theorem to get  $f(t) = 0$  for  $t < 0$ .

Note that if  $F(p)$  is meromorphic then

$$f(t) = \sum \text{residues of } F(p) e^{pt}.$$

It is usually *much* easier to invert Laplace transforms by knowing the answer than by using the inversion formula.

## 5.3 Application to differential equations

### 5.3.1 Ordinary differential equations

We will illustrate this with an example. Suppose we have

$$\ddot{x} - 3\dot{x} + 2x = 4e^t \quad x(0) = -3, \quad \dot{x}(0) = 5. \quad (5.4)$$

Incidentally, Laplace transforms are overkill for this problem; it can be solved easily by using the methods learnt at A-level or in Part 1A.

Generally we use Laplace transforms when we have an initial value problem, not a boundary value problem.

We write  $X(p)$  for the Laplace transform of  $x(t)$  and on transforming (5.4) we get

$$X \{p^2 - 3p + 2\} + 3p - 14 = \frac{4}{p-2}$$

which has the solution

$$X(p) = \frac{4}{(p-2)^2(p-1)} + \frac{14-3p}{(p-2)(p-1)}. \quad (5.5)$$

To invert this we convert to partial fractions:

$$X(p) = \frac{4}{(p-2)^2} + \frac{4}{p-2} - \frac{7}{p-2}$$

which can be inverted (trivially) to get  $x(t) = 4te^{2t} + 4e^{2t} - 7e^t$ .

### Asymptotic behaviour as $t \rightarrow \infty$

Deform the Bromwich contour such that it folds back around the singularity of  $F(p)$  with largest real part (say at  $p = p_0$ ). Then the integral  $\int_{\Gamma} F(p)e^{pt} dp$  is dominated by the neighbourhood of  $p_0$ .

We can approximate this integral by forming an asymptotic expansion of  $F(p)$  about  $p = p_0$ .

In our example (using (5.5)) we see that  $p_0 = 2$  and on writing  $p = 2 + \eta$  we have

$$X(p) = \frac{4}{\eta^2} (1 + \eta)^{-1} + \frac{8-3\eta}{\eta} (1 + \eta)^{-1} \sim \frac{4}{\eta^2} + \frac{4}{\eta} + \text{analytic terms.}$$

Thus

$$\begin{aligned} f(t) &\sim \frac{1}{2\pi i} e^{2t} \int_{\gamma-2-i\infty}^{\gamma+2+i\infty} F(\eta) e^{\eta t} d\eta \\ &\sim e^{2t} (4t + 4). \end{aligned}$$

The  $-7e^t$  term in the exact solution is exponentially smaller than this asymptotic solution and so doesn't feature in the asymptotic expansion.

### Green's functions

For our example we wish to solve

$$\ddot{g} - 3\dot{g} + 2g = \delta(t^+) \quad g(0) = \dot{g}(0) = 0,$$

where  $\delta(t^+)$  is such that  $\int_0^\infty \delta(t^+) dt = 1$ .

Laplace transforming the problem we get  $G(p) = \frac{1}{p-2} - \frac{1}{p-1}$  and so  $g(t) = e^{2t} - e^t$ .

For the general problem

$$\ddot{x} - 3\dot{x} + 2x = f(t) \quad x(0) = x_0, \dot{x}(0) = x_1$$

we find  $X(p) = G(p) \{F(p) + px_0 + x_1 - 3x_0\}$  and so

$$x(t) = g(t) * \{f(t) + x_0\delta'(t^+) - (x_1 - 3x_0)\delta(t^+)\}.$$

### 5.3.2 Partial differential equations

Consider the problem

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial x^2}$$

subject to the boundary conditions  $\phi(x, 0) = 0$ ,  $\phi(x, t) \rightarrow 0$  as  $x \rightarrow \infty$  and  $\phi(0, t) = \phi_0$ . This models (say) the concentration of diffusing salt in a semi-infinite tube.

Laplace transforming the diffusion equation (with respect to  $t$ ) we get (in obvious notation)  $p\tilde{\phi} = \kappa\tilde{\phi}_{xx}$ .

Using the boundary conditions on  $\phi$  we get the solution  $\tilde{\phi} = \frac{\phi_0}{p} e^{-\sqrt{\frac{p}{\kappa}}x}$ .

We can find a lot out about  $\phi$  without doing the inversion. Suppose we wish to evaluate

$$\Phi(t) = \int_0^\infty \phi(x, t) dx$$

which could be the total amount of salt in the tube. Then

$$\tilde{\Phi} = \int_0^\infty \tilde{\phi}(x, p) dx = \phi_0 \sqrt{\frac{\kappa}{p^3}}.$$

We can now inverse transform this to get

$$\Phi(t) = 2\phi_0 \sqrt{\frac{\kappa t}{\pi}}.$$

For the asymptotic behaviour of  $\phi$  as  $t \rightarrow \infty$  we find an asymptotic expansion for  $\tilde{\phi}$  about the largest singularity, which in this case is at  $p = 0$ .

$$\begin{aligned} \tilde{\phi} &\sim \frac{\phi_0}{p} \left( 1 - \sqrt{\frac{p}{\kappa}}x + \dots \right) && \text{thus} \\ \phi &\sim \phi_0 \left( 1 - \frac{x}{\sqrt{\pi\kappa t}} + \dots \right), \end{aligned}$$

valid when  $x \ll \sqrt{\kappa t}$ . This approximation shows that  $\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = -\frac{\phi_0}{\sqrt{\pi\kappa t}}$ .

We now do the full inversion, we see that

$$\begin{aligned} \phi(x, t) &= \frac{\phi_0}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{1}{p} e^{pt - \sqrt{\frac{p}{\kappa}}x} dp \\ &= \frac{\phi_0}{2\pi i} \int_{-\infty}^{(0+)} \frac{1}{p} e^{pt - \sqrt{\frac{p}{\kappa}}x} dp \\ &= \frac{\phi_0}{\pi i} \int_{-\infty}^{\infty} \frac{1}{y} e^{-y^2 t - \frac{iyx}{\sqrt{\kappa}}} dy. \end{aligned}$$

Thus

$$\begin{aligned} \frac{\partial}{\partial x} \phi(x, t) &= -\frac{\phi_0}{\pi\sqrt{\kappa}} \int_{-\infty}^{\infty} e^{-y^2 t - \frac{iyx}{\sqrt{\kappa}}} dy \\ &= -\frac{\phi_0}{\sqrt{\pi\kappa t}} e^{-\frac{x^2}{4\kappa t}}. \end{aligned}$$

Thus

$$\phi(x, t) = \int_x^\infty \frac{\phi_0}{\sqrt{\pi \kappa t}} e^{-\frac{x^2}{4\kappa t}} dx = \frac{2\phi_0}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{\kappa t}}}^\infty e^{-\eta^2} d\eta = \phi_0 \operatorname{erfc} \frac{x}{2\sqrt{\kappa t}}.$$

# References

- Arfken and Weber, *Mathematical Methods for Physicists*, Fourth ed., Academic Press, 1995.

Quite a lot of people sing Arfken's praises; I am not one of them. Although it is useful I think it tries to do too much. If nothing else though, it could be used to kill small mammals.

- E.J. Hinch, *Perturbation Methods*, CUP, 1991.

Useful for the asymptotic expansions part of the course and with enough other theory to make it interesting. Rather more rigorous than the approach taken in this course and much less of a threat to wildlife than Arfken.

- H.A. Priestley, *Introduction to Complex Analysis*, Revised ed., OUP, 1990.

An excellent introduction to contour integrals and the complex analysis needed in this course. Some discussion of transforms but not enough to justify it as a textbook on transforms.

This is an area which appears to be very well served with textbooks. Someone recommend some good ones to me (with *brief* reviews).